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ON SINGULAR MODELS OF A THIN INCLUSION IN A HOMOGENEOUS ELASTIC MEDIUM*

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The equilibrium of an unbounded homogeneous and anisotropic elastic medium containing an inclusion, one of whose characteristic dimensions is much less than the other two is considered. It is assumed that the elastic moduli of the medium and the inclusion differ substantially. The principal terms of the expansion of the elastic fields in the neighbourhood of the thin inclusion are constructed in asymptotic series in small parameters of the problem: the ratio of the characteristic linear dimensions of the inclusion and the ratio of the characteristic values of the elastic moduli of the medium and the inclusion. The problem of constructing the principal terms of these expansions reduces to solving two-dimensional pseudo-differential equations derived by using the procedure of matching the external and internal asymptotic expansions /1, 2/. The results obtained enable two singular models of thin inclusions to be formulated for the cases when their elastic moduli are substantially greater and substantially less than the elastic modulus of the medium. It is shown that one of these models is equivalent to the model of a thin inclusion proposed in /3, 4/.

1. Thin inclusion in a homogeneous elastic medium. In an unbounded homogeneous elastic medium, suppose there is a single inclusion occupying the simply-connected domain V, whose characteristic function is $V(x)$, and $x(x_1, x_2, x_3)$ is a point of the medium. The elastic modulus tensors of the medium and the inclusion are denoted by c_0 and c, respectively. The external stress field $\sigma_0(x)$ (the strains $\varepsilon_0(x)$) is produced by loads applied at infinity.

The stress tensor $\sigma(x)$ and the strain tensor $\varepsilon(x)$ in a medium with an inclusion satisfy the relationships /5/

$$\sigma^{\alpha\beta}(x) = \sigma_0^{\alpha\beta}(x) + \int S^{\alpha\beta\lambda\mu}(x-x') B_{1\lambda\mu\nu\rho}\sigma^{\nu\rho}(x') V(x')\, dx' \tag{1.1}$$

$$\varepsilon_{\alpha\beta}(x) = \varepsilon_{0\alpha\beta}(x) - \int K_{\alpha\beta\lambda\mu}(x-x') c_1^{\lambda\mu\nu\rho}\varepsilon_{\nu\rho}(x') V(x')\, dx' \tag{1.2}$$

$$c_1 = c - c_0, \quad B_1 = B - B_0, \quad B = c^{-1}, \quad B_0 = c_0^{-1}$$

The kernels of the integral operators S and $\mathbf{K}$ in these relationships are expressed in terms of the second derivatives of Green's function $G(x)$ of the Lamé equations for the fundamental medium c_0

$$K_{\alpha\beta\lambda\mu}(x) = -(\nabla_\alpha\nabla_\lambda G_{\beta\mu}(x))_{(\alpha\beta)(\lambda\mu)}, \tag{1.3}$$
$$S^{\alpha\beta\lambda\mu}(x) = c_0^{\alpha\beta\nu\rho} K_{\nu\rho\tau\delta}(x) c_0^{\tau\delta\lambda\mu} - c_0^{\alpha\beta\lambda\mu}\delta(x)$$

Here ∇ is the gradient operator in $\mathbf{R}^3$, and $\delta(x)$ is the three-dimensional delta function. For an arbitrary homogeneous medium, $G(x)$ is an even homogeneous function of degree -1 whose Fourier transform has the form /6/

$$G^*(k) = L^{-1}(k), \quad L^{\alpha\beta}(k) = k_\lambda c_0^{\lambda\alpha\beta\mu} k_\mu \tag{1.4}$$

Hence it follows that $K(x)$ and $S(x)$ are even homogeneous generalized functions of degree -3 whose Fourier transforms are homogeneous functions of zero degree.

Multiplying both sides of (1.1) and (1.2) by $V(x)$, we arrive at systems of elliptical pseudodifferential equations to determine the fields $\sigma^+(x) = \sigma(x) V(x)$ and $\varepsilon^+(x) = \varepsilon(x) V(x)$ within the inclusion. It follows from the properties of the solutions of such equations /7/ that for domains with smooth boundaries, $\sigma^+(x)$ and $\varepsilon^+(x)$ are analytic functions in the domain V for external fields analytic within V. Elastic fields outside the inclusion are restored uniquely by means of known $\sigma^+(x)$ and $\varepsilon^+(x)$ using relationships (1.1) and (1.2).

Now let V be a bounded domain, one of whose characteristic dimensions is small compared with the other two. At each point x of the middle surface Ω of the domain V we select a local Cartesian system of coordinates y_1, y_2, y_3 with axis y_3 directed along the normal $n(x)$

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to Ω. Let $h(x)$ denote the transverse dimension of the domain V along the y_3 axis. The representation $h(x) = \delta_1 l(x)$ holds for the function $h(x)$, where δ_1 is a small dimensionless parameter and $l(x)$ is of the order of the maximum linear dimension of the domain V. We furthermore consider $h(x)$ to be a sufficiently smooth function satisfying the condition $|\partial h(x)| \ll 1$ everywhere in Ω with the exception of a small neighbourhood of the contour Γ, the boundary of Ω. Here ∂ is the gradient operation along Ω

$$\partial_\alpha = \nabla_\alpha - n_\alpha(x)\, n^\beta(x)\, \nabla_\beta, \quad x \in \Omega \tag{1.5}$$

We return to relations (1.1) and (1.2). For a fixed point $x \overline{\in} V$ the kernels $S(x-x')$ and $K(x-x')$ of the integral operators in these relations are smooth bounded functions. Hence, to an accuracy of higher-order infinitesimal components in δ_1, the following equations hold:

$$\sigma(x) = \sigma_0(x) + \int_\Omega S(x-x')\, B_1 \bar{\sigma}^+(x', h)\, d\Omega' \tag{1.6}$$

$$\varepsilon(x) = \varepsilon_0(x) - \int_\Omega K(x-x')\, c_1 \bar{\varepsilon}^+(x', h)\, d\Omega' \tag{1.7}$$

$$\bar{\sigma}^+(x, h) = \int_{-h(x)/2}^{h(x)/2} \sigma^+(x + n(x) y_3)\, dy_3, \quad \bar{\varepsilon}^+(x, h) = \int_{-h(x)/2}^{h(x)/2} \varepsilon^+(x + n(x) y_3)\, dy_3, \quad x \in \Omega \tag{1.8}$$

The terms on the right sides of (1.6) and (1.7) are the principal terms of the asymptotics for the elastic fields outside the fine inclusion. As a rule these terms are of greatest interest for applications (see /3/). The presence of a small parameter in the problem enables us to use perturbation methods (see Sect.4) to construct them. The limit properties of the potentials in (1.6) and (1.7) are first investigated in Sect.2, and the fundamental external limit solutions of the problem are found in Sect.3, i.e. expressions for the elastic fields outside the inclusion are found as the parameter δ_1 tends to zero.

2. The potentials $\sigma_1(x)$ and $\varepsilon_1(x)$.

We introduce the following notation:

$$m(x) = B_1 \bar{\sigma}^+(x, h)$$

where $\bar{\sigma}^+(x, h)$ has the form (1.8). Then the integral components in relationships (1.6) and (1.7) are represented in the form

$$\sigma_1(x) = \int_\Omega S(x-x')\, m(x')\, d\Omega', \quad \varepsilon_1(x) = \int_\Omega K(x-x')\, c_0 m(x')\, d\Omega' \tag{2.1}$$

Assuming $m(x)$ to be as smooth a tensor field in Ω as desired, we investigate the limit values of the potentials $\sigma_1(x)$ and $\varepsilon_1(x)$ as $x \to \Omega$. We will introduce operators for the orthogonal projection on the normal $n(x)$ and the tangent plane to Ω at the point $x \in \Omega$ (π and Θ, respectively)

$$\pi_{\alpha\beta}^{\lambda\mu}(x) = \delta_\alpha^\lambda \delta_\beta^\mu - \Theta_{\alpha\beta}^{\lambda\mu}(x), \quad \Theta_{\alpha\beta}^{\lambda\mu}(x) = (\delta_\alpha^\lambda - n^\lambda(x)\, n_\alpha(x)) \times (\delta_\beta^\mu - n^\mu(x)\, n_\beta(x))_{(\alpha\beta)} \tag{2.2}$$

(δ_β^α is the Kronecker delta). Using these operators, the arbitrary symmetric tensor field $p(x)$ on Ω is expanded into a sum of "normal" $p_1(x)$ and "tangential" $p_2(x)$ components

$$p = p_1 + p_2, \quad p_{1\alpha\beta}(x) = \pi_{\alpha\beta}^{\lambda\mu}(x)\, p_{\lambda\mu}(x), \quad p_{2\alpha\beta}(x) = \Theta_{\alpha\beta}^{\lambda\mu}(x)\, p_{\lambda\mu}(x)$$

The tensor p_1 can be represented in the form

$$p_{1\alpha\beta}(x) = n_{(\alpha}(x)\, b_{\beta)}(x)$$

where $b(x)$ is a certain vector, and the tensor p_2 satisfies the condition

$$n^\alpha(x)\, p_{2\alpha\beta}(x) = 0$$

and, therefore is a tensor of the surface Ω /8/.

Let us represent the density $c_0 m(x)$ of the potential $\varepsilon_1(x)$ in the form of the following sum

$$c_0^{\alpha\beta\lambda\mu} m_{\lambda\mu}(x) = q^{\alpha\beta}(x) + c_0^{\alpha\beta\lambda\mu} n_\lambda(x)\, b_\mu(x) \tag{2.3}$$

where the vector $b(x)$ is a solution of the equation

$$n_\alpha(x)\, c_0^{\alpha\beta\lambda\mu} n_\lambda(x)\, b_\mu(x) = n_\alpha(x)\, c_0^{\alpha\beta\lambda\mu} m_{\lambda\mu}(x)$$

Here the tensor $q(x)$ in (2.3) satisfies the relationships

$$n_\alpha(x)\, q^{\alpha\beta}(x) = 0, \quad \Theta_{\lambda\mu}^{\alpha\beta}(x)\, q^{\lambda\mu}(x) = q^{\alpha\beta}(x) \tag{2.4}$$

and is therefore a tensor of the surface Ω.

We consider the properties of the potential $\varepsilon_1(x)$ with density $c_0 m = q$, which is a tensor of the surface Ω. Using Gauss's theorem for a surface /8/, we have

$$\varepsilon_{1\alpha\beta}(x) = \int_\Omega K_{\alpha\beta\lambda\mu}(x-x')\,\Theta^{\lambda\mu}_{\nu\rho}(x')\,q^{\nu\rho}(x')\,d\Omega' = \qquad (2.5)$$

$$-\nabla_{(\alpha}\int_\Omega G_{\beta)\mu}(x-x')\,\partial_\lambda' q^{\lambda\mu}(x')\,d\Omega' + \oint_\Gamma \nabla_{(\alpha} G_{\beta)\mu}(x-x')\,q^{\lambda\mu}(x')\,e_\lambda(x')\,d\Gamma'$$

where $e(x)$ is the external normal to the contour Γ, that lies in the tangent plane to Ω at the point $x \in \Gamma$. Here (1.3) for $K(x)$ is taken into account, as is also

$$[\nabla_\lambda G_{\beta\mu}(x-x')]\,\Theta^{\lambda\mu}_{\nu\rho}(x')\,q^{\nu\rho}(x') = -\partial_\lambda'[G_{\beta\mu}(x-x')\,q^{\lambda\mu}(x')] + G_{\beta\mu}(x-x')\,\partial_\lambda' q^{\lambda\mu}(x')$$

which follows from (1.5) and (2.2); the prime on the operator ∂ denotes differentiation with respect to x'.

The integral with respect to Γ on the right side of (2.5) is a continuous function for all $x \in \Omega$ ($x \overline{\in} \Gamma$), and the integral with respect to Ω is a symmetrized gradient of the simple field potential of static elasticity theory with density $\partial_\lambda q^{\lambda\mu}(x)$. It is known /9/ that the gradient of a simple layer potential is a regular function in all space and discontinuous on the surface Ω.

Let $\varepsilon_1^+(x)$ and $\varepsilon_1^-(x)$ denote the limit values of the potential $\varepsilon_1(x)$ as the point x tends to Ω from the normal and the opposite sides, respectively. It can be shown that the difference $[\varepsilon_1]$ between the limit values of the potential under consideration is given by

$$[\varepsilon_{1\alpha\beta}(x)] = \varepsilon^+_{1\alpha\beta}(x) - \varepsilon^-_{1\alpha\beta}(x) = n_{(\alpha}(x)\,G^*_{\beta)\mu}(n)\,\partial_\lambda q^{\lambda\mu}(x)$$

where the function $G^*(k)$ has the form (1.4). In the case of an isotropic medium, an analgous relationship for $[\varepsilon_1]$ is obtained in /9/.

It therefore follows that the tangential component $\Theta(x)\varepsilon_1(x)$ of the potential $\varepsilon_1(x)$ is continuous during the passage through Ω. An integral of the type (2.5), representing the value of the function $\Theta(x)\varepsilon_1(x)$ on the surface Ω, diverges formally. The problem of regularizing such integrals was examined in /10/. It can be shown that in this case the regularization formula for the integral mentioned has the form ($x \in \Omega$)

$$\Theta^{\lambda\mu}_{\alpha\beta}(x)\,\varepsilon_{1\lambda\mu}(x) = \int_\Omega U_{\alpha\beta\lambda\mu}(x,x')\,q^{\lambda\mu}(x')\,d\Omega' = \fint_\Omega U_{\alpha\beta\lambda\mu}(x,x')\,[q^{\lambda\mu}(x') - q_0^{\lambda\mu}(x,x')]\,d\Omega' - \qquad (2.6)$$

$$\Theta^{\lambda\mu}_{\alpha\beta}(x)\oint_\Gamma \nabla_\lambda G_{\mu\nu}(x-x')\,q_0^{\nu\rho}(x,x')\,e_\rho(x')\,d\Gamma'$$

$$U_{\alpha\beta\lambda\mu}(x,x') = \Theta^{\nu\rho}_{\alpha\beta}(x)\,K_{\nu\rho\tau\delta}(x-x')\,\Theta^{\tau\delta}_{\lambda\mu}(x') \qquad (2.7)$$

The symbol $\fint$ denotes an integral in the Cauchy principal value sense, and $q_0(x,x')$ is a constant tensor field in Ω. The function $q_0(x,x')$ satisfies the equation $\partial' q_0(x,x') = 0$ and agrees with $q(x)$ for $x' = x$.

Consider a potential $\sigma_1(x)$ of the form (2.1) with density $m(x) = B_0 q(x)$ where $q(x)$ satisfies conditions (2.4)

$$\sigma_1(x) = \int_\Omega S(x-x')\,B_0 q(x')\,d\Omega' \qquad (2.8)$$

It follows from (1.3) that this potential is connected with the potential (2.5) by the relationship

$$\sigma_1^{\alpha\beta}(x) = c_0^{\alpha\beta\lambda\mu}\varepsilon_{1\lambda\mu}(x) - q^{\alpha\beta}(x)\,\delta(\Omega) \qquad (2.9)$$

where $\delta(\Omega)$ is a generalized function concentrated on the surface Ω. It can be seen therefore that the limit values of the potential $\sigma_1(x)$ on Ω agree, apart from the constant c_0, with the limit values of $\varepsilon_1(x)$.

We will now consider a potential $\varepsilon_1(x)$ of the form (2.1) whose density is determined by the second term on the right side of (2.3)

$$\varepsilon_{1\alpha\beta}(x) = \int_\Omega K_{\alpha\beta\lambda\mu}(x-x')\,c_0^{\lambda\mu\nu\rho}n_\nu(x')\,b_\rho(x')\,d\Omega' \qquad (2.10)$$

It follows from (1.3) for $K(x)$ that $\varepsilon_1(x)$ is a symmetrized gradient of a double layer potential of static elasticity theory.

If $b(x)$ in (2.10) is a smooth vector field in Ω, then the potential $\varepsilon_1(x)$ can be represented in the form

$$\varepsilon_{1\alpha\beta}(x) = \varepsilon^r_{1\alpha\beta}(x) + n_{(\alpha}(x)\,b_{\beta)}(x)\,\delta(\Omega) \qquad (2.11)$$

where $\varepsilon_1^r(x)$ is a regular function that is discontinuous on Ω. The limit values of the potential $\varepsilon_1(x)$ are determined by the relationship

$$\varepsilon_{1\alpha\beta}^{\pm}(x) = \int_{\Omega} K_{\alpha\beta\lambda\mu}(x-x') c_0^{\lambda\mu\nu\rho} n_\nu(x') [b_\rho(x') - b_\rho(x)] \, d\Omega' + \tag{2.12}$$

$$R_{\alpha\beta}^{\lambda}(x) b_\lambda(x) \mp \frac{1}{2} \Lambda_{\alpha\beta}^{\lambda\mu}(n) \partial_\lambda b_\mu(x), \quad x \in \Omega$$

$$\Lambda_{\alpha\beta}^{\lambda\mu}(n) = K_{\alpha\beta\nu\rho}^{*}(n) c_0^{\nu\rho\lambda\mu} - I_{\alpha\beta}^{\lambda\mu}, \quad K_{\alpha\beta\lambda\mu}^{*}(k) = (k_\alpha k_\lambda G_{\beta\mu}^{*}(k))_{(\alpha\beta)(\lambda\mu)}$$

Here I is the unit quadrivalent tensor, $G^*(k)$ has the form (1.4), and the function $R(x)$ is represented in the form of an integral over the boundary contour Γ of Ω.

A representation analogous to (2.9)

$$\sigma_1^{\alpha\beta}(x) = c_0^{\alpha\beta\lambda\mu} \varepsilon_{1\lambda\mu}(x) - c_0^{\alpha\beta\lambda\mu} n_\lambda(x) b_\mu(x) \delta(\Omega) \tag{2.13}$$

holds for a potential $\sigma_1(x)$ of the form (2.1) with the density $m(x) = n(x) b(x)$, where $\varepsilon_1(x)$ is the potential (2.10).

Hence, and from (2.11), it follows that the function $\sigma_1(x)$ has no singular components, while its discontinuities on Ω are determined by the relationship

$$[\sigma_1^{\alpha\beta}(x)] = -S^{*\alpha\beta\lambda\mu}(n) \partial_\lambda b_\mu(x), \quad S^{*\alpha\beta\lambda\mu}(n) = c_0^{\alpha\beta\nu\rho} \Lambda_{\nu\rho}^{\lambda\mu}(n)$$

Using this result it can be shown that the vector $n_\beta(x) \sigma_1^{\alpha\beta}(x)$ is continuous on passing through Ω. The value of the function $n_\beta(x) \sigma_1^{\alpha\beta}(x)$ on Ω is given by the formula /10/

$$-n_\beta(x) \sigma_1^{\alpha\beta}(x) = \int_{\Omega} T^{\alpha\beta}(x, x') b_\beta(x') \, d\Omega' = \int_{\Omega} T^{\alpha\beta}(x, x') [b_\beta(x') - b_\beta(x)] \, d\Omega' + \Gamma^{\alpha\beta}(x) b_\beta(x) \tag{2.14}$$

$$T^{\alpha\beta}(x, x') = n_\lambda(x) S^{\lambda\alpha\beta\mu}(x-x') n_\mu(x') \tag{2.15}$$

An expression for the function $\Gamma(x)$ in the form of a contour integral over Γ is given in /10/.

3. The external limit problems.

We will consider solutions of the problem of a thin inclusion as the parameter $\delta_1(h)$ tends to zero. If the inclusion has finite, non-zero elastic moduli, then the elastic fields within the domain V remain bounded in the limit as $\delta_1 \to 0$. It then follows from relations (1.8) that as $h \to 0$ the functions $\bar{\sigma}^+(x, h)$ and $\bar{\varepsilon}^+(x, h)$ vanish. We obtain from (1.6) and (1.7) that the stresses and strains outside the inclusion agree with the unperturbed external fields $\sigma_0(x)$ and $\varepsilon_0(x)$ as the transverse dimension tends to zero.

The need to examine thin inclusions whose elastic moduli differ substantially from the elastic moduli of the medium occurs in a number of applied problems of materials science and fracture mechanics. In the case of inclusions more pliable than the fundamental medium, the tensor c is here represented in the form $c = \delta_2 c_*$, where δ_2 is a small dimensionless parameter while the characteristic values of the tensor component c_* are of the order of the elastic moduli of the medium. If the inclusion is stiffer than the medium, then the tensor B allows of an analgous representation. Since the solution of the elastic problem depends now on two small parameters δ_1 and δ_2, it is best to extract terms of the order of unity compared with δ_1 and δ_2 that yield the principal contribution to the asymptotic of the elastic fields outside the inclusions, in the external expansion. In connection with the problem of constructing these terms, we consider the passages to the limit as the parameters δ_1 and δ_2 tend simultaneously to zero.

First, let the tensor c of the elastic modulus of the inclusion tend to zero together with h. From the relationship

$$B_1\bar{\sigma}^+ = (c^{-1} - c_0^{-1}) c\bar{\varepsilon}^+ = -c_0^{-1} c_1 \bar{\varepsilon}^+$$

it follows that $B_1\bar{\sigma}^+ \to \bar{\varepsilon}^+$ as $c_1 \to -c_0$ $(c \to 0)$. Hence and from (1.6) and (1.7) we obtain that as $h, c \to 0$ $(\delta_1, \delta_2 \to 0)$ the external solution of the problem of a thin inclusion takes the form

$$\sigma(x) = \sigma_0(x) + \int_{\Omega} S(x-x') \bar{\varepsilon}^+(x') \, d\Omega' \tag{3.1}$$

$$\varepsilon(x) = \varepsilon_0(x) + \int_{\Omega} K(x-x') c_0 \bar{\varepsilon}^+(x') \, d\Omega' \tag{3.2}$$

$$\bar{\varepsilon}^+(x) = \lim_{h, c \to 0} \bar{\varepsilon}^+(x, h) \tag{3.3}$$

The function under the limit is defined by the relationship (1.8).

On the other hand, by virtue of the boundedness of the elastic modulus tensor of the inclusion in the limit as $h, c \to 0$ the stress vector $n(x)\sigma(x)$ should remain continuous when passing through the surface Ω. However, the continuity conditions for the displacement vector on Ω can be spoiled because of the disappearance of the elastic modulus of the inclusion. The limit displacement vectors $u(x)$ and stresses on the surface Ω satisfy the relationships

$$[u(x)]\,|_\Omega = b(x), \quad [n(x)\,\sigma(x)]\,|_\Omega = 0 \tag{3.4}$$

where $b(x)$ is a still unknown vector of the displacement field jump in Ω.

It hence follows that the limit strain field $\varepsilon(x)$ should contain a singular component $\delta(\Omega)$ with a coefficient equal to the symmetrized produc of the vectors n and b. We then obtain from (1.8) and (3.3) that the function $\bar{\varepsilon}^+(x)$ has the form

$$\bar{\varepsilon}^+_{\alpha\beta}(x) = n_{(\alpha}(x)\,b_{\beta)}(x) \tag{3.5}$$

Substituting this relationship into (3.1) and (3.2) we arrive at the following expressions for the stress and strain fields in a medium with a limit inhomogeneity:

$$\sigma(x) = \sigma_0(x) + \int_\Omega S(x-x')\,n(x')\,b(x')\,d\Omega' \tag{3.6}$$

$$\varepsilon(x) = \varepsilon_0(x) + \int_\Omega K(x-x')\,c_0 n(x')\,b(x')\,d\Omega' \tag{3.7}$$

The integral components are the potentials $\sigma_1(x)$ and $\varepsilon_1(x)$ of the form (2.1) whose density $m(x)$ agrees with the right side of (3.5). From the properties of the potential $\sigma_1(x)$ mentioned in Sect.2, it follows that the condition (3.4) for the continuity of the vector $n(x)\,\sigma(x)$ on Ω is satisfied automatically for a field $\sigma(x)$ of the form (3.6). But then the equality $(hc \neq 0)$

$$n(x)\,\sigma(x) = h^{-1}(x)\,n(x)\,\bar{\sigma}^+(x, h), \quad x \in \Omega$$

holds to within terms of the order of δ_1, where the function $\bar{\sigma}^+(x, h)$ is defined by the relationship (1.8).

Substituting Hooke's law for an inclusion here

$$\bar{\sigma}^+(x, h) = c\bar{\varepsilon}^+(x, h) \tag{3.8}$$

we obtain the connection between the vector $b(x)$ and the stress vector on the surface

$$n_\beta(x)\,\sigma^{\alpha\beta}(x) = \lambda^{\alpha\beta}(x)\,b_\beta(x), \quad \lambda^{\alpha\beta}(x) = \lim_{h,\,c\to 0} h^{-1}(x)\,n_\lambda(x)\,c^{\lambda\alpha\beta\mu}n_\mu(x) \tag{3.9}$$

by passing to the limit as $h, c \to 0$ and taking (3.5) into account.

Utilizing the expression for the stress tensor (3.6), we hence obtain an equation which the vector field $b(x)$ satisfies on Ω

$$\lambda^{\alpha\beta}(x)\,b_\beta(x) + \int_\Omega T^{\alpha\beta}(x, x')\,b_\beta(x')\,d\Omega' = n_\beta(x)\,\sigma_0^{\alpha\beta}(x), \quad x \in \Omega \tag{3.10}$$

The action of the integral operator T with kernel $T(x, x')$ from (2.15) on the smooth functions $b(x)$ is determined by the right side of (2.14).

Therefore, the limit solution of the problem of a thin inclusion as $h, c \to 0$ has the form (3.6), (3.7), where the vector $b(x)$ is determined from the soluion of (3.10). If $\lambda(x) = 0$, then (3.10) reduces to the equation for the problem of a crack in a homogeneous elastic medium, which is examined in /10, 11/.

Now, let the elastic pliability tensor of the inclusion B tend to zero together with h. Since

$$c_1\bar{\varepsilon}^+ = (B^{-1} - B_0^{-1})B\bar{\sigma}^+ = -B_0^{-1}B_1\bar{\sigma}^+$$

then $c_1\bar{\varepsilon}^+ \to \bar{\sigma}^+$ as $B_1 \to -B_0\ (B \to 0)$.

Hence and from (1.6) and (1.7) we obtain that as $h, B \to 0$ the solution of the problem of a thin inclusion takes the form

$$\sigma(x) = \sigma_0(x) - \int_\Omega S(x-x')\,B_0\bar{\sigma}^+(x')\,d\Omega' \tag{3.11}$$

$$\varepsilon(x) = \varepsilon_0(x) - \int_\Omega K(x-x')\,\bar{\sigma}^+(x')\,d\Omega' \tag{3.12}$$

$$\bar{\sigma}^+(x) = \lim_{h,\,B\to 0} \bar{\sigma}^+(x, h) \tag{3.13}$$

Since the elastic moduli of the inclusion become infinite in the limit, the condition of continuity of the displacement vector on passing through the surface Ω will be satisfied, but the continuity of the stress vector (the local equilibrium condition) may be spoiled. The limit values of the displacement vector and the strain tensor satisfy the following conditions on

$$[u(x)]|_\Omega = 0, \quad [\Theta(x)\varepsilon(x)]|_\Omega = 0 \tag{3.14}$$

The first of these conditions means that the strain field contains no singular component proportional to $\delta(\Omega)$. Hence and from the property (2.11) of the potential $\varepsilon_1(x)$ it follows that the density $\bar{\sigma}^+(x)$ in (3.11) and (3.12) should be a tensor of the surface Ω, i.e., should satisfy the relationships

$$n_\alpha(x)\bar{\sigma}^{+\alpha\beta}(x) = 0, \quad \Theta_{\alpha\beta}^{\lambda\mu}(x)\bar{\sigma}^{+\alpha\beta}(x) = \bar{\sigma}^{+\lambda\mu}(x) \tag{3.15}$$

It therefore follows from the results of Sect.2 that the field (3.12) is a regular function that automatically satisfies the second of conditions (3.14).

Using the continuity of the tangential component of the limit strain tensor on the surface Ω, it can be shown that, apart from terms of the order of δ_1, the equation

$$\Theta(x)\varepsilon(x) = h^{-1}(x)\Theta(x)\bar{\varepsilon}^+(x, h)$$

is satisfied, where the function $\bar{\varepsilon}^+(x, h)$ has the form (1.8).

Expressing $\bar{\varepsilon}^+(x, h)$ in terms of $\bar{\sigma}^+(x, h)$ by using the Hooke's law for an inclusion (3.8), taking account of (3.15), and passing to the limit as $h, B \to 0$, we obtain in the preceding relationship

$$\Theta(x)\varepsilon(x) = \mu(x)\bar{\sigma}^+(x), \quad \mu_{\alpha\beta\lambda\mu}(x) = \lim_{h,\, B\to 0} h^{-1}(x)\Theta_{\alpha\beta}^{\nu\rho}(x)B_{\nu\rho\tau\delta}\Theta_{\lambda\mu}^{\tau\delta}(x) \tag{3.16}$$

Substituting expression (3.12) for $\varepsilon(x)$ here, we arrive at an equation for the field $\bar{\sigma}^+(x)$ on Ω

$$\mu_{\alpha\beta\lambda\mu}(x)\bar{\sigma}^{+\lambda\mu}(x) + \int_\Omega U_{\alpha\beta\lambda\mu}(x, x')\bar{\sigma}^{+\lambda\mu}(x')\,d\Omega' = \Theta_{\alpha\beta}^{\lambda\mu}(x)\varepsilon_{0\lambda\mu}(x), \quad x \in \Omega \tag{3.17}$$

The action of an operator U with kernel $U(x,x')$ of the form (2.7) on smooth functions $\bar{\sigma}^+(x)$ is governed by the right side of relationship (2.6). If $\mu = 0$, then (3.17) goes over into an equation for an absolutely rigid surface (membrane) in a homogeneous elastic medium.

It can be shown that the operators $\mathbf{T}$ and $\mathbf{U}$ in (3.10) and (3.17) are elliptic pseudo-differential operators whose principal homogeneous symbols are homogeneous functions of the first degree /10/. By continuity, the operators $\mathbf{T}$ and $\mathbf{U}$ are extended to the class of Sobolev-Slobodetskii generalized functions on Ω /7/. The symbols of the pseudo-differential operators in (3.10) and (3.17) differ from the symbols $\mathbf{T}$ and $\mathbf{U}$ by positive-definite components ($\lambda(x)$ and $\mu(x)$, respectively) and hence are also elliptic. The conditions for equations of the type (3.10) and (3.17) to be solvable are investigated in /7/.

The preceding consideration shows that the external solution of the problem of a thin inclusion obtained as the parameters δ_1 and δ_2 tend to zero is not defined uniquely and depends on the limit of the ratio δ_1/δ_2. The magnitude of this ratio determines the functions $\lambda(x)$ and $\mu(x)$ in (3.10) and (3.17). For a unique assignment of these functions, we use the procedure of matching the external and internal asymptotic expansions of the solutions in small parameters of the problem /1, 2/.

4. The internal limit problem and matching procedure. We take an arbitrary point x on the surface Ω $(x \in \Gamma)$ as the centre of a local y_1, y_2, y_3 coordinate system. We define the natural internal variables of the problem of a thin inclusion by the relationships /1/

$$\xi_i = h^{-1}(x)y_i, \quad i = 1, 2, 3$$

Letting the parameter $\delta_1(h)$ tend to zero, we arrive at the internal limit problem which (in the coordinates ξ_i) is a problem on the equilibrium of a homogeneous elastic medium containing an inclusion in the form of a plane layer of unit thickness in the domain $|\xi_3| \leqslant 1/2$.

Let $\sigma_e(x)$ and $\varepsilon_e(x)$ denote the stress and strain fields corresponding to the solution of the external limit problem. Using the method of matching the external and internal asymptotic expansions /1, 2/, we take the elastic fields on the interface between the medium and inclusion in the internal limit problem equal to the limit values of the fields $\sigma_e(x)$ and $\varepsilon_e(x)$ at the point $x \in \Omega$

$$\lim_{\xi_3\to\pm 1/2} \sigma(\xi) = \sigma_e^\pm(x), \quad \lim_{\xi_3\to\pm 1/2} \varepsilon(\xi) = \varepsilon_e^\pm(x) \tag{4.1}$$

It is here assumed that the point $\xi(\xi_1, \xi_2, \xi_3)$ tends to the plane $\xi_3 = 1/2$ or $\xi_3 = -1/2$ while remaining outside the layer.

Let the elastic moduli tensor c of the inclusion be small compared with the elastic moduli tensor of the medium ($cc_0^{-1} = O(\delta_2)$). It is natural to select the zeroth approximations $\sigma_e^\circ(x)$ and $\varepsilon_e^\circ(x)$ for the external solutions $\sigma_e(x)$ and $\varepsilon_e(x)$ in a form agreeing with (3.6) and (3.7), and corresponding to the limit of the external solution as $\delta_1, \delta_2 \to 0$

$$\sigma_e^\circ(x) = \sigma_0(x) + \int_\Omega S(x-x')\, n(x')\, b_0(x')\, d\Omega' \tag{4.2}$$

$$\varepsilon_e^\circ(x) = \varepsilon_0(x) + \int_\Omega K(x-x')\, c_0 n(x')\, b_0(x')\, d\Omega' \tag{4.3}$$

Here the vector $b_0(x)$ is the solution of (3.10) in which we consider $\lambda(x)$ to be still an arbitrary smooth function of order c/h.

From the properties mentioned in Sect.2 for the potential on the right side of (4.2), it follows that the stress vector $n(x)\sigma_e^\circ(x)$ is continuous in Ω. Hence, its limit values $n\sigma_e^{\circ+}$ and $n\sigma_e^{\circ-}$ at the point $x \in \Omega$ agree and have the form

$$n(x)\sigma_e^{0+}(x) = n(x)\sigma_e^{\circ-}(x) = \lambda(x)\, b_0(x) \tag{4.4}$$

The relationship (3.9) which the external solution $\sigma_e^\circ(x)$ satisfies, is taken into account here.

Using (2.12), we write an expression for the limit values of the tangential component of the tensor $\varepsilon_e^\circ(x)$ on the surface Ω

$$\Theta(x)\, \varepsilon_e^{\circ\pm}(x) = \Theta(x)\, D(x) \pm \Theta(x)\, \partial b_0(x) \tag{4.5}$$

$$D(x) = \varepsilon_0(x) + \int_\Omega K(x-x')\, c_0 n(x')\, [b_0(x') - b_0(x)]\, d\Omega' + R(x)\, b_0(x)$$

It is known /12/ that on the interface between the medium and the inclusion the stress vector and the tangential component of the strain tensor are continuous. Consequently, the conditions (4.1) for $\sigma_e = \sigma_e^\circ$, $\varepsilon_e = \varepsilon_e^\circ$ can be satisfied if the stresses $\sigma^+(\xi)$ and strains $\varepsilon^+(\xi)$ within the layer are defined by the relationships

$$n(x)\,\sigma^+(\xi) = \lambda(x)\, b_0(x), \quad \sigma^+(\xi) = c\varepsilon^+(\xi) \tag{4.6}$$
$$\Theta(x)\,\varepsilon^+(\xi) = \Theta(x)\, D(x) + 2\xi_3\Theta(x)\,\partial b_0(x)$$

Note that conditions (4.1) do not define the elastic fields within the layer uniquely. As will be shown below, when constructing the principal terms of the asymptotic expansion under consideration, it is possible to limit oneself to the simplest linear approximation (4.6) of the elastic fields within the layer.

Changing to the external variable y_i in (4.6) and substituting the result into (1.8), we find an expression for the integral characteristic $\bar\varepsilon^+(x, h)$ in the zeroth approximation

$$\pi_{\alpha\beta}^{\lambda\mu}(x)\,\bar\varepsilon_{\lambda\mu}^+(x, h) = n_{(\alpha}(x)\, b_{1\beta)}(x), \quad \Theta_{\alpha\beta}^{\lambda\mu}(x)\,\bar\varepsilon_{\lambda\mu}^+(x, h) = O(\delta_1) \tag{4.7}$$
$$b_{1\alpha}(x) = g_\alpha^{\ \beta}(x)\, b_{0\beta}(x) + O(\delta_2), \quad g(x) = h(x)\, d^{-1}(x)\,\lambda(x)$$
$$d^{\alpha\beta}(x) = n_\lambda(x)\, c^{\lambda\alpha\beta\mu} n_\mu(x), \quad g(x) = O(\delta_1/\delta_2)$$

The function $\bar\sigma^+(x, h)$ is expressed in terms of $\bar\varepsilon^+(x, h)$ by using (3.8). Here and henceforth it is assumed that $\delta_1/\delta_2 = O(1)$.

We substitute $\bar\varepsilon^+(x, h)$ from (4.7) into the right side of (1.7) and denote the result by $\varepsilon_i(x)$. The field $\varepsilon_i(x)$ is the external limit of the internal solution (see /1/). Extracting terms of the order of δ_1/δ_2 in the expression for $\varepsilon_i(x)$, we will have

$$\varepsilon_i(x) = \varepsilon_0(x) + \int_\Omega K(x-x')\, c_0 n(x')\, g(x')\, b_0(x')\, d\Omega' + O(\delta_1, \delta_2)$$

Comparing this expression with the right side of (4.3), we obtain that the principal terms of the external limit of the internal solution $\varepsilon_i(x)$ merge with the external solution $\varepsilon_e^\circ(x)$ provided that

$$g_\beta^{\ \alpha}(x) = \delta_\beta^{\ \alpha} \quad \text{or} \quad \lambda(x) = h^{-1}(x)\, d(x) \tag{4.8}$$

It hence follows that the external expansions of elastic fields outside a thin inclusion have the form

$$\sigma(x) = \sigma_e^\circ(x) + O(\delta_1, \delta_2), \quad \varepsilon(x) = \varepsilon_e^\circ(x) + O(\delta_1, \delta_2) \tag{4.9}$$

where the functions $\sigma_e^\circ(x)$ and $\varepsilon_e^\circ(x)$ are defined according to the relationships (4.2) and (4.3) and the vector $b_0(x)$ therein is the solution of (4.10) for $\lambda(x)$ in the form (4.8).

Now, let the elastic compliance tensor of the material of the inclusion be small compared with the elastic compliance tensor of the medium ($BB_0^{-1} = O(\delta_2)$). By matching the external and internal asymptotic expansions, we select the zeroth approximation for the external solutions σ_e and ε_e in the form (3.11) and (3.12) corresponding to the limit of the external solution as $\delta_1, \delta_2 \to 0$

$$\sigma_e^\circ(x) = \sigma_0(x) - \int_\Omega S(x - x') B_0 \bar{\sigma}_0^+(x')\, d\Omega' \tag{4.10}$$

$$\varepsilon_e^\circ(x) = \varepsilon_0(x) - \int_\Omega K(x - x') \bar{\sigma}_0^+(x')\, d\Omega'$$

The density $\bar{\sigma}_0^+(x)$ is determined from (3.17) in which $\mu(x)$ is still an arbitrary smooth function of order B/h.

The internal limit problem has the same meaning as in the preceding case. To solve the internal problem under conditions (4.1), expression (4.10) should now be substituted for the external solutions. It can be shown as before that the external limit of the internal solution matches the external solution (4.10) if the density $\bar{\sigma}_0^+(x)$ of the potentials in these relationships satisfies (3.17) for $\mu(x)$ in the form

$$\mu_{\alpha\beta\lambda\mu}(x) = h^{-1}(x)\, \Theta_{\alpha\beta}^{\nu\rho}(x)\, B_{\nu\rho\tau\delta} \Theta_{\lambda\mu}^{\tau\delta}(x) \tag{4.11}$$

The functions $\sigma_e^\circ(x)$ and $\varepsilon_e^\circ(x)$ of the form (4.10) are here the principal terms of the external expansions of the elastic fields outside the thin inclusion and, apart from components of the order of δ_1 and δ_2, approximate these fields everywhere with the exception, generally, of a small neighbourhood of the boundary contour Γ.

5. Singular models of thin inclusions. In considering thin inclusions in a homogeneous elastic medium, it is best in a number of cases to replace the three-dimensional inclusion by an equivalent two-dimensional singular model. It is proposed /3, 4/ to replace the problem of a thin inclusion by a boundary value problem of elasticity theory for a homogeneous medium c_0, and to select the conditions on the middle surface of the inclusion Ω modeling the presence of an inclusion in the form

$$[n\sigma]|_\Omega = 0, \quad [u]|_\Omega = b, \quad n\sigma|_\Omega = \lambda b \tag{5.1}$$

where the tensor λ has the form (4.8). The first of these conditions is called the equilibrium condition in /3/, and the last is called Hooke's law for the inclusion. It is assumed that the solution of the boundary value problem satisfying the given conditions at infinity is a good approximation of the asymptotic for the elastic layer outside the thin inclusion.

The boundary conditions (5.1) evidently agree in form with conditions (3.4), (3.9). Consequently, the solution of such a boundary value problem has the form (3.6), (3.7), where the vector $b(x)$ is determined from (3.10).

The connection between the solution of the model problem under consideration and the exact asymptotic form of the elastic field outside the thin inclusion is essentially given in Sect.4. By virtue of the estimate (4.9), this solution will aproximate the asymptotic form of the exact solution all the more, the smaller the magnitude of the relative transverse dimension of the inclusion δ_1 and the ratio δ_2 of the characteristic elastic modulus of the inclusion to the characteristic modulus of the medium. It is natural to call such inclusions crack-like. The effect of hindering the strain on the inclusion is not taken into account when replacing them by a singular model with conditions (5.1) on Ω, whereupon the tangential component of the strain tensor may turn out to be discontinuous on the surface Ω.

To construct the asymptotic form of the elastic fields outside a thin inclusion whose elastic moduli are substantially greater than the moduli of the medium, it is natural to use a two-dimensional singular model with the conditons (3.14) and (3.16) on Ω where the former can be called the strain compatibility condition, and the latter can be called Hooke's law for the inclusion. The solution of this problem has the form (3.11), (3.12), where the function $\bar{\sigma}^+(x)$ is determined from (3.17) in which $\mu(x)$ has the form (4.11). As is shown in Sect. 4, a singular model of a rigid inclusion with conditions (3.14) and (3.16) enables us to describe the asymptotic form of the elastic field outside the inclusion more exactly, the smaller the parameters δ_1 and δ_2, where δ_2 is the ratio of the characteristic elastic modulus of the medium to the characteristic elastic modulus of the inclusion. Only the effect constraining the strain on the inclusion, which plays the principal part in the case of rigid inclusions, is taken into account in replacing the real inclusion by such a singular model.

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A METHOD OF MAKING IMPEDANCE MEASUREMENTS OF THE VISCOELASTIC PROPERTIES OF A MEDIUM BY OSCILLATING PLATES AND SHELLS*

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A method of measuring the viscoelastic properties of a homogeneous medium bounded by plates and shells is presented, based on processing observations of arbitrary oscillations.

The impedance measurements an attempt is usually made to use the simplest forms of oscillations and obtain one-dimensional motion. However if the region in which oscillations are excited is of limited size this gives rise to difficulties due to diffraction or edge effects, and the excitation of modes of oscillation that are not used in the measurements. An increase in the dimensions of the excitation region in order to reduce the influence of these effects usually requires an increase in the stiffness, and hence also in the mass of the probes, which reduces the measurement sensitivity, and makes them virtually impossible at high frequencies.

A method is propsoed in the present paper of processing the observed arbitrary oscillations of plates and shells, which enable us to obtain the same results and formulas for the simplest modes of oscillation including one-dimensional modes. This is also feasible in cases for which this realization is practically impossible, which enables the range of impedance frequency measurements to be extended. The viscoelastic properties of the medium are determined in terms of displacements and stresses on the plate or shell surfaces for arbitrarily small oscillations.

Let the oscillations of a homogeneous plate or shell with bounding surfaces S_1 and S_2 be used for impedance measurements, where the surface S_2 is in contact with the viscoelastic medium being investigated. The oscillations observed on the surface S_1 and their properties are used to determine the properties of the medium. Such plates or shells can be, for example, the walls of apparatus, autoclaves, containers, or pipelines, the values of underground structures, or natural objects. We shall restrict our consideration to the simplest configurations of plates and shells.

The equations of small oscillations of a plate (shell) and medium have the form /1/

$$k_j^{-2}\,\mathrm{grad\,div}\,\mathbf{u}_j - \varkappa_j^{-2}\,\mathrm{rot\,rot}\,\mathbf{u}_j + \mathbf{u}_j = 0 \tag{1}$$

$$k_j = \omega/c_j,\quad \varkappa_j = \omega/v_j,\quad c_j = \sqrt{(\lambda_j + 2\mu_j)/\rho_j},\quad v_j = \sqrt{\mu_j/\rho_j}$$

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